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Exercises for Advanced Physics 2, 2019/20 term 4

Exercise Set 6

(Due date: Tuesday, Febuary 4, 2020)

Exercise 6 (**Interchange of spins**) (10 points)

We consider the Pauli spin operator $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ with

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

that obey the relations

$$\begin{aligned} \sigma_i^2 &= 1\,, \quad i = x, y, z \\ \sigma_i \sigma_j &= i\sigma_k\,, \quad (i,j,k) \in \big\{(x,y,z),(z,x,y),(y,z,x)\big\} \\ [\sigma_i, \sigma_j]_+ &= \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\mathbb{1}\,, \qquad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{1}$$

$\vec{\sigma}^{(i)}$, $i = 1, 2$ is the spin operator for the particle i.

(a) Prove $\big(\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}\big)^2 = 3\mathbb{1} - 2\big(\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}\big)$

(b) Show that the operator

$$Q_{12} = \frac{1}{2}\big(\mathbb{1} + \vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}\big)$$

interchanges the spins of particles 1 and 2:

$$\begin{aligned} Q_{12}\vec{\sigma}^{(1)}Q_{12}^{-1} &= \vec{\sigma}^{(2)} \\ Q_{12}\vec{\sigma}^{(2)}Q_{12}^{-1} &= \vec{\sigma}^{(1)} \end{aligned} \tag{2}$$