Okayama University
Faculty of Science

Research Institute for Interdisciplinary Science
Prof. Harald Jeschke
Tutor: Makoto Shimizu

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Exercises for Advanced Physics 2, 2019/20 term 4

Exercise Set 4
(Due date: Tuesday, January 21, 2020)

Exercise 4 (**Spin operator commutation relations**) (10 points)

We introduced the following **anticommutation** relations for the creation operators $c^\dagger_{i\sigma}$ (annihilation operators $c_{i\sigma}$) of an electron with spin σ ($\sigma = \uparrow, \downarrow$) at lattice site $\vec{R}_i$:

$$
\begin{aligned}
&\left[c_{i\sigma}, c_{j\sigma'}\right]_+ = \left[c^\dagger_{i\sigma}, c^\dagger_{j\sigma'}\right]_+ = 0 \\
&\left[c_{i\sigma}, c^\dagger_{j\sigma'}\right]_+ = \delta_{ij}\delta_{\sigma\sigma'}
\end{aligned}
\tag{1}
$$

(a) Explain why the spin operators can be defined by

$$
\begin{aligned}
S^z_i &= \frac{\hbar}{2}\left(n_{i\uparrow} - n_{i\downarrow}\right) \quad \text{with } n_{i\sigma} = c^\dagger_{i\sigma}c_{i\sigma} \\
S^+_i &= S^x_i + iS^y_i = \hbar c^\dagger_{i\uparrow}c_{i\downarrow} \\
S^-_i &= S^x_i - iS^y_i = \hbar c^\dagger_{i\downarrow}c_{i\uparrow}
\end{aligned}
\tag{2}
$$

(b) Verify the **commutation** relations for the spin operators:

(i) $\left[S^+_i, S^-_i\right] = 2\hbar S^z_i$

(ii) $\left[S^z_i, S^+_i\right] = \hbar S^+_i$

(iii) $\left[S^z_i, S^-_i\right] = -\hbar S^-_i$

(iv) If there is one spin per lattice site ($S = \frac{1}{2}$): $\vec{S}^2_i = \hbar^2 S(S+1)$