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Exercises for Advanced Physics 3, 2018 term 4

Exercise Set 4

(Due date: Tuesday, January 22, 2019)

Exercise 4 (**Spin operator commutation relations**) (10 points)

We introduced the following **anticommutation** relations for the creation operators $c_{i\sigma}^\dagger$ (annihilation operators $c_{i\sigma}$) of an electron with spin σ ($\sigma = \uparrow, \downarrow$) at lattice site $\vec{R}_i$:

$$\begin{aligned} &[c_{i\sigma}, c_{j\sigma'}]_+ = [c_{i\sigma}^\dagger, c_{j\sigma'}^\dagger]_+ = 0 \\ &[c_{i\sigma}, c_{j\sigma'}^\dagger]_+ = \delta_{ij}\delta_{\sigma\sigma'} \end{aligned} \tag{1}$$

(a) Explain why the spin operators can be defined by

$$\begin{aligned} S_i^z &= \frac{\hbar}{2}(n_{i\uparrow} - n_{i\downarrow}) \quad \text{with } n_{i\sigma} = c_{i\sigma}^\dagger c_{i\sigma} \\ S_i^+ &= S_i^x + iS_i^y = \hbar c_{i\uparrow}^\dagger c_{i\downarrow} \\ S_i^- &= S_i^x - iS_i^y = \hbar c_{i\downarrow}^\dagger c_{i\uparrow} \end{aligned} \tag{2}$$

(b) Verify the **commutation** relations for the spin operators:

(i) $[S_i^+, S_i^-] = 2\hbar S_i^z$

(ii) $[S_i^z, S_i^+] = \hbar S_i^+$

(iii) $[S_i^z, S_i^-] = -\hbar S_i^-$

(iv) If there is one spin per lattice site ($S = \frac{1}{2}$): $\vec{S}_i^2 = \hbar^2 S(S+1)$