Unkonventionelle Supraleitung

Serie 4

Verteilung: 22.November Abgabe: 29.November

4.1 Generalized BCS theory. Consider the Hamiltonian given by Eq. (2.15) or (93) in the theory lecture notes:

$$H = \sum_k C_k^\dagger \check{\varepsilon}_k C_k,$$

with

$$C_k = \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \\ c_{-k\uparrow}^\dagger \\ c_{-k\downarrow}^\dagger \end{pmatrix} \qquad \text{and} \qquad \check{\varepsilon}_k = \frac{1}{2}\begin{pmatrix} \xi_k \hat{\sigma}_0 & \hat{\Delta}_k \\ \hat{\Delta}_k^\dagger & -\xi_k \hat{\sigma}_0 \end{pmatrix},$$

where $\hat{\sigma}_0$ is the 2×2 unit matrix. $\xi_{-k} = \xi_k$. $C_k^\dagger = (c_{k\uparrow}^\dagger, c_{k\downarrow}^\dagger, c_{-k\uparrow}, c_{-k\downarrow})$. $\hat{\Delta}_k$ possesses a symmetry: $\hat{\Delta}_{-k} = -\hat{\Delta}_k^T$, ($\hat{\bullet}^T$ means a transpose matrix). Here, we have omitted an unimportant classical-number term K in the Hamiltonian of Eq. (2.15) or (93). As a notation, *"hat"* $\hat{\bullet}$ denotes the 2×2 matrix in the spin space, and *"check"* $\check{\bullet}$ denotes the 4×4 matrix composed of the 2×2 particle-hole (Nambu) space and the 2×2 spin space.

The above Hamiltonian is known to be diagonalized as

$$\begin{aligned} H &= \sum_k \left(C_k^\dagger \check{U}_k\right)\left(\check{U}_k^\dagger \check{\varepsilon}_k \check{U}_k\right)\left(\check{U}_k^\dagger C_k\right) \\ &= \sum_k A_k^\dagger \check{E}_k A_k, \end{aligned}$$

with

$$\check{E}_k = \frac{1}{2}\begin{pmatrix} \hat{E}_k & 0 \\ 0 & -\hat{E}_{-k} \end{pmatrix} \qquad \text{and} \qquad \hat{E}_k = \begin{pmatrix} E_{k,+} & 0 \\ 0 & E_{k,-} \end{pmatrix},$$

by the Bogoliubov transformation:

$$A_k = \check{U}_k^\dagger C_k \qquad \text{with} \qquad \check{U}_k = \begin{pmatrix} \hat{u}_k & \hat{v}_k \\ \hat{v}_{-k}^* & \hat{u}_{-k}^* \end{pmatrix}.$$

Here, $\check{U}_k^\dagger \check{U}_k = \check{U}_k \check{U}_k^\dagger = \check{1}$ with the 4×4 unit matrix $\check{1}$.

Show the followings for the unitary states $\Big($i.e., for $\hat{\Delta}_k \hat{\Delta}_k^\dagger = \hat{\Delta}_k^\dagger \hat{\Delta}_k = |\Delta_k|^2 \hat{\sigma}_0$ with $|\Delta_k|^2 \equiv \frac{1}{2}\mathrm{Tr}[\hat{\Delta}_k \hat{\Delta}_k^\dagger]$ and $|\Delta_{-k}| = |\Delta_k|$ $\Big)$:

$$E_{k,+}^2 = E_{k,-}^2 = \xi_k^2 + |\Delta_k|^2 \quad (\equiv E_k^2), \tag{1}$$

and

$$\hat{u}_k = \frac{(E_k + \xi_k)\hat{\sigma}_0}{\sqrt{2E_k(E_k+\xi_k)}}, \qquad \hat{v}_k = \frac{-\hat{\Delta}_k}{\sqrt{2E_k(E_k+\xi_k)}}. \tag{2}$$

Here, we assume $E_k > 0$. In your calculation, assume $\hat{u}_k = u_k \hat{\sigma}_0$ and u_k is real.

4.2 Consider the gap equation given in Eq. (2.22) or (100) in the theory lecture notes

$$\Delta_{k,s_1s_2} \quad = \quad -\sum_{k',s_3s_4} V_{k,k';s_1s_2s_3s_4} \frac{\Delta_{k',s_4s_3}}{2E_{k'}} \tanh\Big(\frac{E_{k'}}{2k_{\mathrm{B}}T}\Big),$$

and the pairing interaction in Eq. (2.23) or (101) having the form

$$V_{k,k';s_1s_2s_3s_4} = J^0_{k,k'}\sigma^0_{s_1s_4}\sigma^0_{s_2s_3} + J_{k,k'}\vec{\sigma}_{s_1s_4}\cdot\vec{\sigma}_{s_2s_3}.$$

Here, $\hat{\sigma}^0$ is the 2×2 unit matrix and $\hat{\vec{\sigma}} = (\hat{\sigma}^x, \hat{\sigma}^y, \hat{\sigma}^z)$ are the Pauli matrices. $A_{ss'}$ denotes the ss' matrix element of $\hat{A}$, $(s, s' = \{\uparrow, \downarrow\})$. $\hat{A}$ stands for $\hat{\Delta}_k$, $\hat{\sigma}^0$, and $\hat{\vec{\sigma}}$.

Show that the gap equation can be written as

$$\Psi_k \quad = \quad -\sum_{k'}(J^0_{k,k'} - 3J_{k,k'})\frac{\Psi_{k'}}{2E_{k'}} \tanh\Big(\frac{E_{k'}}{2k_{\mathrm{B}}T}\Big)$$

for singlet states $\hat{\Delta}_k = \Psi_k i\hat{\sigma}^y$, and

$$\vec{d}_k \quad = \quad -\sum_{k'}(J^0_{k,k'} + J_{k,k'})\frac{\vec{d}_{k'}}{2E_{k'}} \tanh\Big(\frac{E_{k'}}{2k_{\mathrm{B}}T}\Big)$$

for triplet states $\hat{\Delta}_k = \vec{d}_k \cdot \hat{\vec{\sigma}} i\hat{\sigma}^y$.